

L_h^2 -DOMAINS OF HOLOMORPHY AND THE BERGMAN KERNEL

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ABSTRACT. We give a characterization of L_h^2 -domains of holomorphy with the help of the boundary behavior of the Bergman kernel and geometric properties of the boundary, respectively.

For $\lambda_0 \in \mathbb{C}$, $r > 0$ we define $\Delta(\lambda_0, r) := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$. We also put $E := \Delta(0, 1)$. Moreover, the set of all plurisubharmonic (respectively, subharmonic) functions on an open set $D \subset \mathbb{C}^n$ is denoted by $\text{PSH}(D)$ (respectively, $\text{SH}(D)$). We allow the (pluri)subharmonic functions to be equal identically to $-\infty$ on connected components of D .

Following [Kli] for a domain $D \subset \mathbb{C}^n$ define

$$g_D(p, z) := \sup\{u(z)\}, \quad p, z \in D,$$

where the supremum is taken over all negative $u \in \text{PSH}(D)$ such that $u(\cdot) - \log \|\cdot - p\|$ is bounded from above near p . We call the function $g_D(p, \cdot)$ *the pluricomplex Green function* (with the logarithmic pole at p). We also denote

$$A_D(p; X) := \limsup_{\lambda \rightarrow 0} \frac{\exp(g_D(p, p + \lambda z))}{|\lambda|}, \quad p \in D, X \in \mathbb{C}^n.$$

Following [Jar-Pfl] the function A_D is *the Azukawa pseudometric*.

For a boundary point w of a bounded domain $D \subset \mathbb{C}$ we introduce the notion of regularity. Namely, we say that D is *regular* at w if there exist a neighborhood U of w and a subharmonic function u on $U \cap D$ with $u < 0$ on $U \cap D$ and $\lim_{U \cap D \ni \lambda \rightarrow w} u(\lambda) = 0$.

A set $P \subset \mathbb{C}^n$ is called *pluripolar* if for any point $z \in P$ there exist a connected neighborhood $U = U(z)$ and a function $u \in \text{PSH}(U)$, $u \not\equiv -\infty$, such that $P \cap U \subset \{z \in U : u(z) = -\infty\}$. In case $n = 1$ we call such a set P *polar*. It is well known (cf. [Kli], Josefson theorem) that a set $P \subset \mathbb{C}^n$ is pluripolar if and only if there is a function $u \in \text{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $P \subset \{z \in \mathbb{C}^n : u(z) = -\infty\}$.

A bounded domain $D \subset \mathbb{C}^n$ is said to be *hyperconvex* if there exists a negative and continuous plurisubharmonic exhaustion function of D .

Key words and phrases. Bergman kernel, L_h^2 -domain of holomorphy, (pluri)polar sets.

The second author was supported by the KBN grant No. 2 P03A 017 14. While working on the paper the second author was at the stay at the Carl von Ossietzky Universität Oldenburg (Germany) supported by DAAD.

Denote the class of square integrable holomorphic functions on an open set D by $L_h^2(D)$. $L_h^2(D)$ is a Hilbert space with the standard scalar product induced from $L^2(D)$. Let us recall the definition of *the Bergman kernel*:

$$K_D(z) := \sup \left\{ \frac{|f(z)|^2}{\|f\|_{L_h^2(D)}^2} : f \not\equiv 0, f \in L_h^2(D) \right\}.$$

If D is a bounded domain then $\log K_D$ is smooth and strictly plurisubharmonic. Therefore, for a bounded domain D one may define *the Bergman metric* β_D :

$$\beta_D(z; X) := \sqrt{\sum_{j,k=1}^n \frac{\partial^2 \log K_D(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k}, \quad z \in D, X \in \mathbb{C}^n.$$

With the help of the Bergman metric we obtain:

$$b_D(w, z) := \inf \{ L_{\beta_D}(\alpha) \}, \quad w, z \in D$$

where $L_{\beta_D}(\alpha) = \int_0^1 \beta_D(\alpha(t); \alpha'(t)) dt$ and the infimum is taken over all piecewise C^1 -curves $[0, 1] \mapsto D$. We call b_D the Bergman distance. If (D, b_D) is a complete metric space we say that D is *Bergman complete*.

A domain $D \subset \mathbb{C}^n$ is called a *(an L_h^2 -)domain of holomorphy* if there are no domains $D_0, D_1 \subset \mathbb{C}^n$ with $\emptyset \neq D_0 \subset D_1 \cap D$, $D_1 \not\subset D$ such that for any $f \in \mathcal{O}(D)$ ($f \in L_h^2(D)$) there exists an $\tilde{f} \in \mathcal{O}(D_1)$ with $\tilde{f} = f$ on D_0 .

Let us recall several results concerning the above mentioned notions, which show a close relationship between the theory of square integrable holomorphic functions and that of the pluripotential theory.

For a bounded pseudoconvex domain D consider the following properties:

- (1) D is hyperconvex,
- (2) for any $w \in \partial D$, $\lim_{D \ni z \rightarrow w} K_D(z) = \infty$,
- (3) D is Bergman complete,
- (4) D is an L_h^2 -domain of holomorphy.

All the relations between the properties (1)–(4) are known. Namely, $((1) \implies (2))$ (see [Ohs 1]), $((1) \implies (3))$ (see [Blo-Pfl], [Her]), $((3) \implies (4))$. The implication $((2) \implies (1))$ does not hold in general (take the Hartogs triangle in \mathbb{C}^2 or consider some one dimensional Zalcman-type domains – see [Ohs 1]). The one-dimensional counterexample to the implication $((3) \implies (1))$ is given in [Chen 1]. Recall that any bounded pseudoconvex fat domain is an L_h^2 -domain of holomorphy (see [Pfl]). Thus the Hartogs triangle is an L_h^2 -domain of holomorphy in \mathbb{C}^2 which is not Bergman complete. Moreover, there exists also a fat domain in the complex plane that is not Bergman complete (see [Jar-Pfl-Zwo]). Thus, the implication $((4) \implies (3))$ does not hold even for fat pseudoconvex domains. In dimension one the implication $((2) \implies (3))$ holds (see [Chen 2]) but in higher dimension this is no longer the case (take the Hartogs triangle once more). As far as the implication $((3) \implies (2))$ is concerned one may find a counterexample already in dimension one (see [Zwo 2]).

Let us have a closer look at the last example. The domains being counterexamples belong to the following class of domains:

$$D := E \setminus \left(\bigcup_{j=1}^{\infty} \bar{\Delta}(z_j, r_j) \cup \{0\} \right),$$

where $z_j \rightarrow 0$, $r_j > 0$, $\bar{\Delta}(z_j, r_j) \subset E \setminus \{0\}$, $\bar{\Delta}(z_j, r_j) \cap \bar{\Delta}(z_k, r_k) = \emptyset$, $j \neq k$. It is easy to see that for any $w \in \partial D$, $w \neq 0$ we have $\lim_{D \ni z \rightarrow w} K_D(z) = \infty$. The point is that the sequences can be chosen so that $\liminf_{D \ni z \rightarrow 0} K_D(z) < \infty$ and the domain is still Bergman complete. On the other hand one may easily see that $\limsup_{z \rightarrow 0} K_D(z) = \infty$. So the natural problem arises whether one may construct an example of a Bergman complete domain such that for some $w \in \partial D$ we have $\limsup_{z \rightarrow w} K_D(z) < \infty$. Below we show that this is impossible. Let us write down explicitly the condition we are interested in (as some kind of appendix to properties ((1)–(4))):

$$(5) \quad \text{for any } w \in \partial D \text{ we have } \limsup_{D \ni z \rightarrow w} K_D(z) = \infty.$$

The main aim of this paper is to present the following characterizations of L_h^2 -domains of holomorphy.

Theorem 1. *Let D be a bounded pseudoconvex domain in \mathbb{C}^n . Then (4) is equivalent to (5).*

Making use of Theorem 1 and a result of A. Sadullaev we also get the following characterization of bounded L_h^2 -domains of holomorphy.

Theorem 2. *Let D be a bounded pseudoconvex domain. Then D is an L_h^2 -domain of holomorphy if and only if for any $w \in \partial D$ and for any neighborhood U of w the set $U \setminus D$ is not pluripolar.*

Before proving Theorem 1 let us recall some properties of the notions that we have just defined and that we need in the sequel.

First we list a number of properties of polar sets that we shall use (see [Ran], [Con]).

Let D be an open set in \mathbb{C} and let $K \subset D$ be a polar set relatively closed in D . Then

- if D is additionally connected then so is $D \setminus K$,
- for any $\lambda \in D$ and for any $0 < s$ with $\Delta(\lambda, s) \subset\subset D$ there is an $s < r$ with $\Delta(\lambda, r) \subset\subset D$ and $\partial\Delta(\lambda, r) \cap K = \emptyset$,
- for any $f \in L_h^2(D \setminus K)$ there is an $\tilde{f} \in O(D)$ such that $\tilde{f}|_{D \setminus K} = f$.

There is also a precise description of L_h^2 -domains of holomorphy in \mathbb{C} .

Theorem 3 (see [Con], Theorem 9.9, p. 351). *Let D be a bounded domain in \mathbb{C} and let $z \in \partial D$. Then there is an open neighborhood U of z such that any $f \in L_h^2(D)$ extends holomorphically to $D \cup U$ if and only if there is a neighborhood V of z such that the set $V \setminus D$ is polar.*

One may easily get from Theorem 3 the following description of L_h^2 -domains of holomorphy in \mathbb{C} .

Theorem 4. *Let D be a bounded domain in \mathbb{C} . Then D is an L_h^2 -domain of holomorphy iff for any $w \in \partial D$ and for any neighborhood U of w the set $U \setminus D$ is not polar.*

Note that Theorem 2 is the exact more dimensional counterpart of Theorem 4.

Let us recall now some basic properties of regular points and the Green function. For a domain $D \subset \mathbb{C}^n$ we have $g_D(p, \cdot) \in \text{PSH}(D)$, $g_D(p, \cdot) < 0$. A bounded domain D is hyperconvex iff $g_D(p, \cdot)$ is a continuous exhaustive function of D .

In the case of bounded planar domains it is well-known that the Green function is symmetric (as the function of two variables) and $g_D(p, \cdot)$ is harmonic on $D \setminus \{p\}$. Moreover, a point $w \in \partial D$ is regular iff for some (any) $p \in D$ $g_D(p, \lambda) \rightarrow 0$ as $D \ni \lambda \rightarrow w$. Consequently, a bounded domain $D \subset \mathbb{C}$ is hyperconvex iff any point from its boundary is regular. The set of irregular points of any bounded domain in \mathbb{C} is polar.

Below we shall need some estimate for the Bergman kernel in the one-dimensional case that will enable us to prove Theorem 1 in dimension one.

Theorem 5 (see [Ohs 2]). *Let D be a domain in \mathbb{C} . Then there is a positive constant C such that*

$$\sqrt{K_D(z)} \geq CA_D(z; 1), \quad z \in D.$$

Our first aim is to obtain the following exhaustion property of the Bergman kernel at regular points.

Proposition 6. *Let D be a bounded domain in \mathbb{C} . Assume that $w \in \partial D$ is a regular point. Then*

$$K_D(z) \rightarrow \infty$$

as $D \ni z \rightarrow w$.

Proof. In view of Theorem 5 it is sufficient to show that

$$(6) \quad r(p) \rightarrow 0 \text{ as } p \rightarrow w,$$

where $r := r(p) := \text{diam } D(p)$, $D(p) := \{z \in D : g_D(p, z) < -1\}$. In fact, assuming the last property we get (see [Zwo 1])

$$A_D(p; 1) = eA_{D(p)}(p; 1) \geq eA_{\Delta(p, r)}(p; 1) = \frac{e}{r} \rightarrow \infty \text{ as } p \rightarrow w.$$

Suppose that (6) does not hold. Then one easily finds an $\epsilon > 0$, sequences $D \ni p_\nu \rightarrow w$ and $D \ni z_\nu \rightarrow z \in \bar{D}$ such that $|p_\nu - z_\nu| \geq \epsilon$ and $g_D(p_\nu, z_\nu) < -1$. Taking $\tilde{D} := D \cup V$, where V is some small disc around z such that $w \notin \bar{V}$, we get $g_D(p_\nu, z_\nu) \geq g_{\tilde{D}}(p_\nu, z_\nu)$ and $z \in \tilde{D}$. In other words, it is sufficient to show that $g_{\tilde{D}}(p_\nu, z_\nu) \rightarrow 0$. But because of the pointwise convergence of $g_{\tilde{D}}(p_\nu, \cdot) = g_{\tilde{D}}(\cdot, p_\nu)$ to 0 (as $\nu \rightarrow \infty$), the harmonicity of $g_{\tilde{D}}(p_\nu, \cdot)$ near z and the Vitali theorem, we conclude that $g_{\tilde{D}}(p_\nu, \cdot)$ tends uniformly to 0 on some neighborhood of z , which finishes the proof. \square

Remark 7. In view of property (6) it follows from the estimates in [Die-Her] that for any bounded domain in \mathbb{C} the convergence $\beta_D(z; 1) \rightarrow \infty$ as $z \rightarrow w \in \partial D$ holds for any regular point $w \in \partial D$.

Lemma 8. *Let D be a bounded domain in \mathbb{C} , $w \in \partial D$. Then the following conditions are equivalent:*

$$(7) \quad \limsup_{D \ni z \rightarrow w} K_D(z) < \infty,$$

(8) *there is an open neighborhood U of w such that the set $U \setminus D$ is polar.*

Proof. Let us make at first a general remark. Namely, the condition $U \setminus D$ is polar is equivalent to the condition $U \cap \partial D$ is polar.

((8) \implies (7)). If U satisfies (8) then without loss of generality one may assume that $K := U \cap \partial D \subset\subset U$. So there is a domain \tilde{D} with $D = \tilde{D} \setminus K$, $w \in \tilde{D}$, where K is a compact polar set. Then

$$L_h^2(D) = L_h^2(\tilde{D})|_D$$

and, consequently,

$$K_D = K_{\tilde{D}|_D},$$

which implies (7).

((7) \implies (8)). Suppose that for any neighborhood U of w the set $U \cap \partial D$ is not polar. Then there is a sequence $w_\nu \rightarrow w$, $w_\nu \in \partial D$, such that D is regular at w_ν . In view of Proposition 6 we have $K_D(z) \rightarrow \infty$ as $D \ni z \rightarrow w_\nu$, which easily finishes the proof. \square

Lemma 9. *Let D be a domain in \mathbb{C}^n , $n \geq 2$. Fix $0 < r < t$. For any $z' \in \mathbb{C}^{n-1}$ define $A(z') := \{z_n \in tE : (z', z_n) \in D\} := tE \setminus K(z')$. Assume that $K(0')$ is polar and there is a neighborhood $0' \in V$ such that for almost any $z' \in V$ (with respect to the $(2n-2)$ -dimensional Lebesgue measure) the set $K(z')$ is polar. Then there is a neighborhood $0' \in V' \subset V$ such that for any $f \in L_h^2(D)$ there exists a function $F \in \mathcal{O}(V' \times rE)$ with $F = f$ on $(V' \times rE) \cap D$.*

Proof. Because $K(0')$ is polar there is an s with $0 < r < s < t$ such that $K(0') \cap \partial(sE) = \emptyset$. Then there is a neighborhood $0' \in V' \subset V$ such that for any $\zeta' \in V'$ we have $K(\zeta') \cap \partial(sE) = \emptyset$.

Define

$$F(\zeta', z_n) := \frac{1}{2\pi i} \int_{\partial(sE)} \frac{f(\zeta', \lambda) d\lambda}{\lambda - z_n}, \quad (\zeta', z_n) \in V' \times sE.$$

Then F is a holomorphic function on $V' \times sE$.

On the other hand because of the square integrability of f , the Fubini Theorem and because of the assumptions of the lemma, for almost all $\zeta' \in V'$ (with respect to the $(2n-2)$ -dimensional Lebesgue measure) the function $f(\zeta', \cdot) \in L_h^2(tE \setminus K(\zeta'))$ and $K(z')$ is polar. Since closed polar sets are removable for L_h^2 -functions, for almost all $\zeta' \in V'$ the function $f(\zeta', \cdot)$ extends to a holomorphic function on tE . So the Cauchy formula applies and we obtain the equality $f(\zeta', z_n) = F(\zeta', z_n)$, $(\zeta', z_n) \in (V' \times sE) \cap D$ for almost all $\zeta' \in V'$. Since equality holds on a dense subset of $(V' \times sE) \cap D$, the equality holds on the whole set, which finishes the proof. \square

Before we start the proof of Theorem 1 let us formulate, in the form that we need, the most powerful tool we shall use, namely the Ohsawa-Takegoshi extension theorem.

Theorem 10 (see [Ohs-Tak]). *Let D be a bounded pseudoconvex domain in \mathbb{C}^n and let L be a complex line. Then there is a constant $C > 0$ such that for any $f \in L_h^2(D \cap L)$ there is an $F \in L_h^2(D)$ with $\|F\|_{L_h^2(D)} \leq C\|f\|_{L_h^2(D \cap L)}$ and $F|_{D \cap L} = f$.*

Note that Theorem 10 directly leads to the following inequality for the Bergman kernel:

$$K_{D \cap L}(z) \leq C^2 K_D(z), \quad z \in D \cap L.$$

This inequality will be often used in our next considerations. Note only that the set $D \cap L$ on the left-hand side of the above inequality is open (as a subset of \mathbb{C}) but not necessarily connected.

We now prove our main result.

Proof of Theorem 1. First note that the result for $n = 1$ follows from Theorem 4 and Lemma 8, so assume that $n \geq 2$.

((5) \implies (4)). Suppose that D is not an L_h^2 -domain of holomorphy. Then there are a polydisc $P \subset D$ with $\partial P \cap \partial D \neq \emptyset$ and a polydisc $P \subset \subset \tilde{P}$ such that for every function $f \in L_h^2(D)$ there is a function $\hat{f} \in H^\infty(\tilde{P})$ with the property $f = \hat{f}$ on P .

We claim that for any $z \in P$ and for any complex line L passing through z we have

$$L \cap D \cap \tilde{P} = (L \cap \tilde{P}) \setminus K(z), \quad \text{where } K(z) \text{ is a polar set.}$$

Suppose that $L \cap D \cap \tilde{P} = (L \cap \tilde{P}) \setminus K(z)$, where $K(z)$ is not a polar set. Then choose a compact non-polar set $K' \subset K(z) \subset (L \cap \tilde{P}) \setminus D$ such that $V_0 = L \setminus \hat{K}'$ (\hat{K}' denotes the polynomial hull of K') contains $L \cap P$. Then there is a function $f \in L_h^2(V_0)$ which does not extend holomorphically through \hat{K}' (cf. Theorem 3). Let $\{V_j\}_{j=1}^N$, where $0 \leq N \leq \infty$ be the family of bounded components of $L \setminus K'$. We, additionally, let f be identically 0 on $\bigcup_{j=1}^N V_j$.

In view of the Ohsawa-Takegoshi extension theorem there exists an $F \in L_h^2(D)$ such that $F|_{L \cap D} = f|_{L \cap D}$. But then there is a function $\hat{F} \in H^\infty(\tilde{P})$ such that $\hat{F}|_P = F|_P$. Consequently, $\hat{F}|_{L \cap \tilde{P}}$ is a holomorphic extension of $f|_{L \setminus \hat{K}'}$ through \hat{K}' – contradiction.

It follows from the above claim that $\tilde{P} \cap D$ is connected. Consequently, for any function $f \in L_h^2(D)$ its (uniquely determined) extension $\hat{f} \in H^\infty(\tilde{P})$ satisfies the equality $f = \hat{f}$ on $D \cap \tilde{P}$.

Consider the following normed space

$$A := \{(f, \hat{f}) : f \in L_h^2(D)\} \subset L_h^2(D) \times H^\infty(\tilde{P})$$

with the norm $\|(f, \hat{f})\| := \|f\|_{L_h^2(D)} + \|\hat{f}\|_{H^\infty(\tilde{P})}$. It is easily seen that A is a Banach space. Consider the following mapping:

$$\pi : A \ni (f, \hat{f}) \mapsto f \in L_h^2(D).$$

Then π is a one-to-one surjective continuous linear mapping. Hence, in view of the Banach open mapping theorem, π^{-1} is a continuous linear mapping. In other words, there is a constant $C > 1$ such that

$$\|(f, \hat{f})\| \leq C\|f\|, \quad \text{where } f \in L_h^2(D);$$

in particular, $\|\hat{f}\|_{H^\infty(\tilde{P})} \leq C\|f\|_{L_h^2(D)}$. Consequently,

$$\sup_{z \in \tilde{P} \cap D} K_D(z) = \sup\left\{ \frac{|f(z)|^2}{\|f\|_{L_h^2(D)}^2} : z \in \tilde{P} \cap D, f \not\equiv 0, f \in L_h^2(D) \right\} \leq C^2,$$

which contradicts (5) for any $w \in \partial P \cap \partial D \neq \emptyset$.

((4) \implies (5)). Fix $w \in \partial D$.

Let us first consider the case $w \notin \text{int}(\bar{D})$. Then there is a sequence $z_\nu \rightarrow w$, $z_\nu \notin \bar{D}$. Let B_ν be the largest open ball centered at z_ν disjoint from \bar{D} . Choose $w_\nu \in \partial B_\nu \cap \partial D$. Obviously, $w_\nu \rightarrow w$. Note that for any ν , D satisfies at w_ν 'the outer cone condition' (see [Pfl]). Therefore, for any ν we have $\lim_{D \ni z \rightarrow w_\nu} K_D(z) = \infty$ (see [Pfl]), which easily implies (5).

Assume now that $w \in \text{int}(\bar{D})$. Suppose that (5) does not hold at w . Then there is a polydisc P with centre at w such that $\sup\{K_D(z) : z \in D \cap P\} < \infty$. Without loss of generality we may assume that $P \subset \subset \text{int}(\bar{D})$. Consider any complex line L intersecting P . We claim that $L \cap P \cap D$ is equal to $(L \cap P) \setminus K$, where K is a polar set or $K = L \cap P$. In fact if this were not the case then $\sup_{z \in L \cap P \cap D} K_{L \cap D}(z) = \infty$ (the Bergman kernel is here understood as that of a one-dimensional set) (use Lemma 8) and, consequently, in view of the Ohsawa-Takegoshi extension Theorem we get $\sup_{z \in L \cap P \cap D} K_D(z) = \infty$ – contradiction.

Note that there is a complex line L passing through w such that $L \cap P \cap D$ is not empty. Assume that $w = 0$. Making a linear change of coordinates and shrinking P if necessary we may assume that $P = E^n$ and that $\{\lambda \in E : (0, \dots, 0, \lambda) \in D\}$ is not empty.

Therefore, the assumptions of Lemma 9 are satisfied (with some neighborhood $V \subset E^{n-1}$ of $0' \in \mathbb{C}^{n-1}$) and there is a neighborhood $0' \in V' \subset E^{n-1}$ such that for any $f \in L_h^2(D)$ there is a function $F \in \mathcal{O}(V' \times \frac{1}{2}E)$ with $F = f$ on $(V' \times \frac{1}{2}E) \cap D$ – contradiction. \square

Proof of Theorem 2. Because of Theorem 4 we may assume that $n \geq 2$.

(\implies). Suppose that for some $w \in \partial D$ there is a polydisc P such that $P \setminus D$ is pluripolar. Let $u \in \text{PSH}(P)$ be such that $u \not\equiv -\infty$ and $P \setminus D \subset \{u = -\infty\}$. Take a nonempty open set $U \subset D \cap P$ and consider all complex lines connecting w with some point from U . It is easy to see that there is a complex line L such that $u \not\equiv -\infty$ on $L \cap P$. Assume that $w = 0$. Making linear change of coordinates and shrinking P , if necessary, we may assume that $P = E^n$ and $\{z_n \in E : (0', z_n) \notin D\}$ is polar. Because of the local integrability of u , for almost any $z' \in E^{n-1}$ (with respect to the $(2n-2)$ -dimensional Lebesgue measure) the function $u(z', \cdot)$ is not identically equal to $-\infty$ on E . Consequently, for almost any $z' \in E^{n-1}$ the set $\{z_n \in E : (z', z_n) \notin D\}$ is polar. Applying Lemma 9 we obtain the existence of an open set $0 \in Q$ such that for any $f \in L_h^2(D)$ there exists an $F \in \mathcal{O}(Q)$ with $f = F$ on $D \cap Q$ – contradiction.

(\Leftarrow). Suppose that the implication does not hold, so in view of Theorem 1 there is a $w \in \partial D$ such that $\limsup_{D \ni z \rightarrow w} K_D(z) < \infty$. In other words there is a polydisc P with centre at w such that $\sup_{z \in D \cap P} K_D(z) < \infty$.

First note that for any complex line L with $L \cap P \neq \emptyset$ we have $L \cap P \cap D = \emptyset$ or $L \cap P \cap D = (L \cap P) \setminus K$, where K is a polar set. Actually, if there were L such that $L \cap P \cap D = (L \cap P) \setminus K$, where $K \neq L \cap P$ and K is not polar, then for some $U \subset \subset L \cap P$, $\sup_{z \in U \cap D} K_{D \cap L}(z) = \infty$ (use Lemma 8). Therefore, in view of the Ohsawa-Takegoshi theorem, $\sup_{z \in U \cap D} K_D(z) = \infty$ – contradiction.

Consequently, one may apply a result of A. Sadullaev (see [Sad 2] and also [Sad 1]) to get that the set $P \setminus D$ is pluripolar – contradiction. \square

It follows from the reasoning in proofs of Theorems 1 and 2 that the following more dimensional counterpart of Lemma 8 holds

Lemma 11. *Let D be a bounded pseudoconvex domain and let $w \in \partial D$. Then $\limsup_{D \ni z \rightarrow w} K_D(z) < \infty$ if and only if for any neighborhood U of w the set $U \setminus D$ is pluripolar.*

The known examples of L_h^2 -domains of holomorphy include, among others, bounded pseudoconvex fat domains and bounded pseudoconvex balanced domains. The characterization of L_h^2 -domains of holomorphy given by us yields many examples of such domains. Below we give one example of a new class of domains having this property.

For a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ we define the following Hartogs domain with m -dimensional balanced fibers:

$$G_D := \{(w, z) \in \mathbb{C}^{n+m} : H(z, w) < 1\},$$

where $\log H$ is plurisubharmonic on $D \times \mathbb{C}^m$, $H(z, \lambda w) = |\lambda|H(z, w)$, $(z, w) \in D \times \mathbb{C}^m$, $\lambda \in \mathbb{C}$, and G_D is bounded (i.e. $H(z, w) \geq C\|w\|$ for some $C > 0$, $(z, w) \in D \times \mathbb{C}^m$). In such a situation G_D is a bounded pseudoconvex domain.

Proposition 12. *Let D be a bounded L_h^2 -domain of holomorphy. Then G_D (with notation as above) is an L_h^2 -domain of holomorphy.*

Proof. Let us take $(z^0, w^0) \in \partial G_D$. If $z^0 \in D$ then $\lim_{G_D \ni (z, w) \rightarrow (z^0, w^0)} K_{G_D}(z, w) = \infty$ (use Theorem 3.1(i) from [Jar-Pfl-Zwo]).

Assume now that $z^0 \in \partial D$. Let V be any neighborhood of (z^0, w^0) . In view of Lemma 11 and Theorem 1 it is sufficient to show that $V \setminus G_D$ is not pluripolar. Without loss of generality we may assume that $V = V_1 \times V_2 \subset \mathbb{C}^{n+m}$. Because D is an L_h^2 -domain of holomorphy Theorem 2 applies and $V_1 \setminus D$ is not pluripolar. Since $V \setminus G_D \supset (V_1 \setminus D) \times V_2$ and the latter set is not pluripolar, the proof is finished. \square

Acknowledgment. The authors would like to thank A. Edigarian for drawing their attention to a result of A. Sadullaev.

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